Plans
- Friday class: will be a video to watch — on the lecture notes page.
- If you do better on the final exam than your lowest test grade, the final exam grade will also replace that test grade.
- Review for final exam will be out by the end of the week. 40% newest stuff, 60% old stuff.

$\boxed{\text{Ex}}$ Let R be the relation on $\mathbb{R}$ defined by

$(a,b) \in R \iff b = a^2$. Find $R^3 = R \circ R \circ R$

Review: $R \circ S = \{(a,c) : aSb \text{ and } bRc \text{ for some } b\}$

$R \circ R \circ R = ?$

$R \circ R = \{(a,c) : aRb \text{ and } bRc \text{ for some } b.$

$R \circ R \circ R = \{(a,p) : a\, R\circ R\, c \text{ and } cRp \text{ for some } c\}$

$= \{(a,p) : aRb \text{ and } bRc \text{ and } cRp \text{ for some } b \text{ and some } c\}$

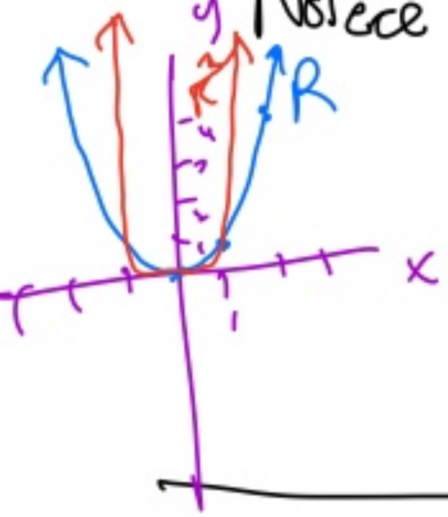
R is a relation on $\mathbb{R}$

$$R^3 = R \circ R \circ R = \{(a,p) : \overbrace{aRb}^{(a,b)\in R} \text{ and } \overbrace{bRc}^{(b,c)\in R} \text{ and } \overbrace{cRp}^{(c,p)\in R} \text{ for some } b \text{ and some } c\}.$$

$$= \{(a,p) : b = a^2 \text{ and } c = b^2 \text{ and } p = c^2 \text{ for some } b, c \in \mathbb{R}\}$$

$c = b^2 = (a^2)^2 = a^4, \quad p = c^2 = (a^4)^2 = a^8$

$= \{(a,p) : p = a^8\} = \{(a, a^8) : a \in \mathbb{R}\}$



Notice R is just the graph of the function

$$f(x) = x^2$$

R^3 is the graph of the function

$$g(x) = x^8 = f(f(f(x))),$$
$$= (f \circ f \circ f)(x).$$

Thus, the ways relations are composed is consistent with how functions are composed.

<u>Example</u> Let T be the relation on $\mathbb{R}$ defined by $T = \{(x,y) : y < x < 2y\}$.

Let's compute T^2.

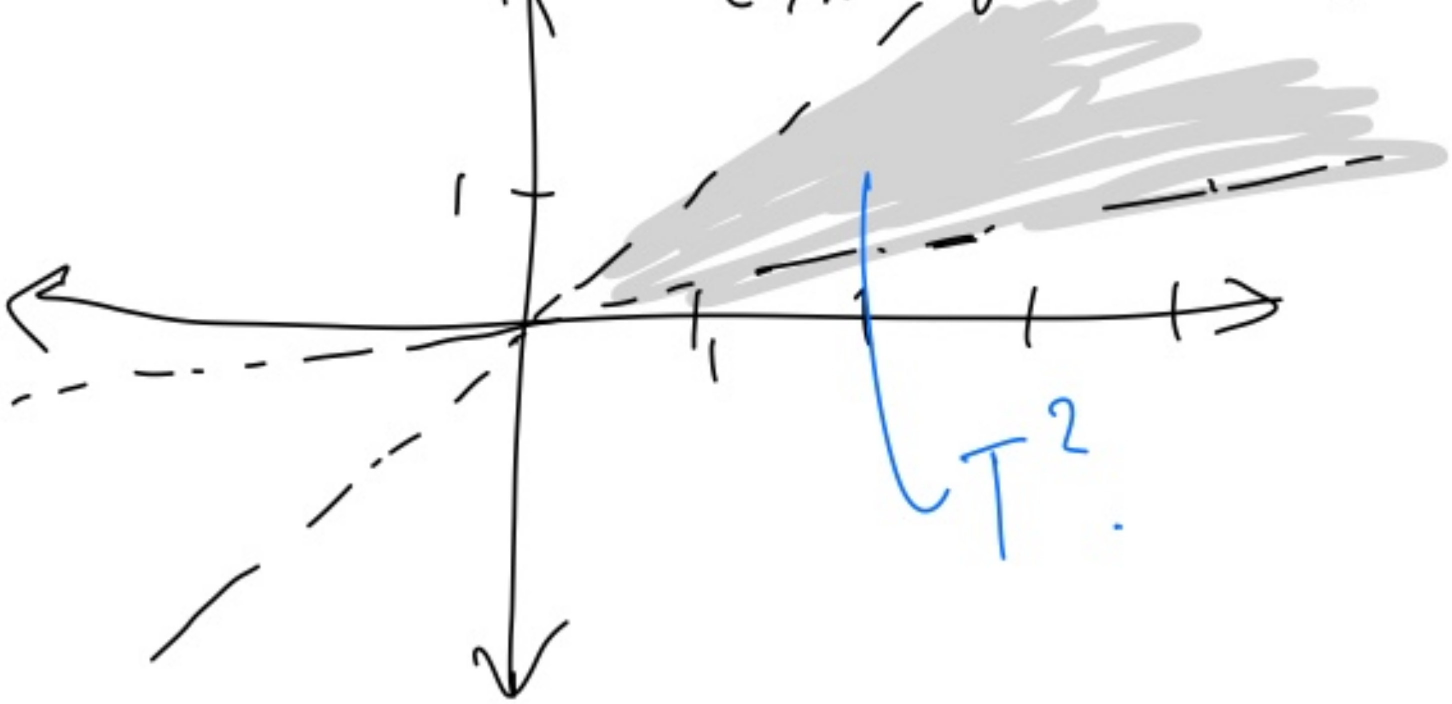
$$T^2 = \{(a,c) : aTb \text{ and } bTc \text{ for some } b \in \mathbb{R}\}$$

$$T^2 = T \circ T = \{(a,c) : b < a < 2b \text{ and } c < b < 2c \text{ for some } b \in \mathbb{R}\}$$

$$= \{(a,c) : b < a \text{ and } \boxed{a < 2b} \text{ and } c < b \text{ and } \boxed{b < 2c} \text{ for some } b \in \mathbb{R}\}.$$

$$= \{(a,c) : c < a \text{ and } \boxed{a < 4c}\}$$

$$\{(x,y) : y < x \text{ and } x < 4y\}$$



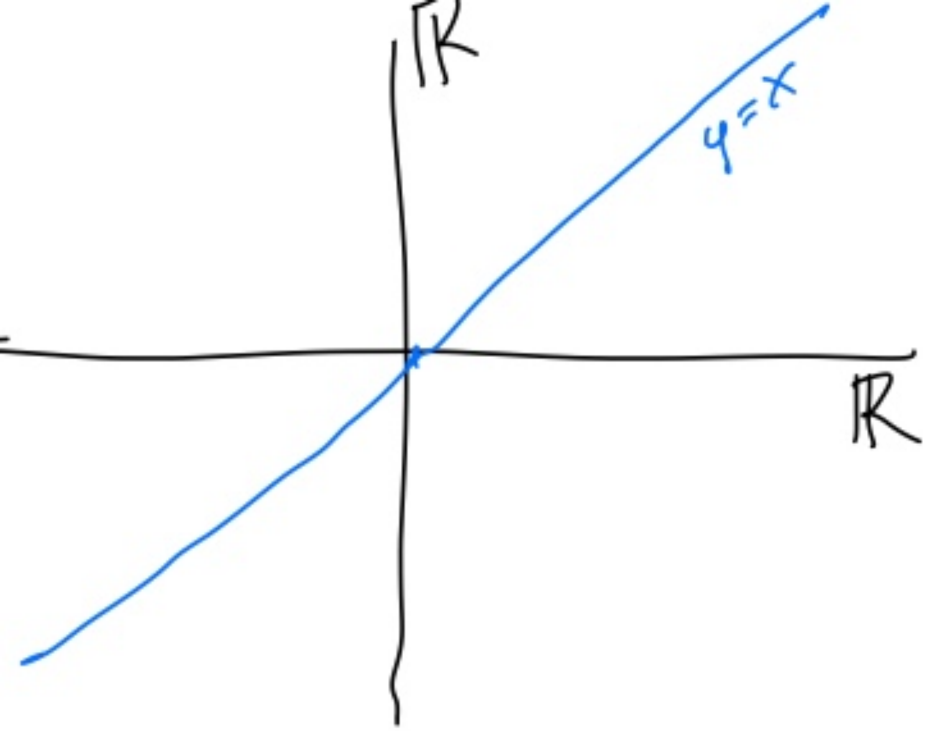
Note we can do anything with relations that can be done with sets. ie If R_1, R_2 are two relations on a set A, that means

$$R_1 \subseteq A \times A, \quad R_2 \subseteq A \times A.$$

So we can make new relations with intersections $R_1 \cap R_2$, unions $R_1 \cup R_2$, etc.

Finite Relations.

Question: Let S be a nonempty relation on $\mathbb{R}$ that is very small as a set, but so that it is an equivalence relation. What could S be?



Could it be one point?
No. Because of the reflexive property —
$(x,x) \in$ relation S $\forall x \in \mathbb{R}$.

So every point on the line $y=x$ must be part of any equivalence relation on $\mathbb{R}$.

Could the relation be $S = \{(x,y) : y = x \in \mathbb{R}\}$?

It satisfies the reflexive property, because $(x,x) \in S$ $\forall x \in \mathbb{R}$.

It also satisfies symmetry, since If $(x,y) \in S$, then $y = x$, so also $(y,x) = (x,y) \in S$.

It also satisfies the transitive property, because if $(a,b) \in S$

are $(b,c) \in S$ for some b, then

$a=b$ and $b=c \Rightarrow a=c$

so $(a,c) \in S$ also.

So $S = \{(x,x) : x \in \mathbb{R}\}$ is the smallest possible equivalence relation on $\mathbb{R}$.

If $A = \{a_1, a_2, \dots, a_n\}$ is a finite set of points, there can smaller equiv. relations.

eg. If $A = \{1\}$, then the relation $\{(1,1)\}$ on A is an equivalence relation.

(called the TCU Horned Frog relation).

If $A = \{1, 2\}$, what are the possible equivalence relations in $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$??

Answers: (1) $\{(1,1), (2,2)\}$

(2) $\{(1,1), (2,2), (2,1), (1,2)\}$

Question: If $B = \{1,2,3\}$, what are the possible equivalence relations? on B